Intrinsic Slepian-Bangs type formula for parameters on LGs with unknown measurement noise variance

Samy Labsir
IPSA/TéSA
Toulouse, France

Alexandre Renaux
University of Paris-Saclay
Gif-sur-Yvette, France

Jordi Vilà-Valls
DEOS/NAVIR^{2}ES
ISAE-SUPAERO
Toulouse, France

Eric Chaumette
DEOS/NAVIR^{2}ES
ISAE-SUPAERO
Toulouse, France

***Abstract*—Intrinsic lower bounds on the intrinsic mean square error are of major importance to characterize the best achievable estimation performance of any unbiased estimator on smooth manifold as Lie group (LG). When the parameter is described by a LG model and the observation noise is with unknown variance, it is also necessary to determine a bound both on the LG parameter and this variance. In this communication, we propose an intrinsic generic Fisher information matrix taking into account this problem. To achieve that, we derive an intrinsic Slepian-Bangs formula on the LG product of the unknown parameter of interest and the LG of positive scalar values in which the variance intrinsically lies. The proposed bound is validated on a Gaussian observation model for unknown parameter lying to $SE(3)$ and variance noise on $\mathbb{R}^+$.**

I. Introduction

It is well-known that the performance of Euclidean estimators can be characterized through lower bounds on the mean square error (MSE) [1]. The most popular is the Cramér-Rao bound (CRB) [2], [3], mainly due to its tractable form, and because it gives a precise estimation of the maximum likelihood estimator (MLE) for Euclidean Gaussian model, under some regularity conditions for large number of observations. [4], [5].

In the last decade there has been an increasing interest in the derivation of intrinsic lower bounds [6]–[10], where the parameters of interest live in a Lie group (LG), a manifold of interest, which appears in many signal processing applications. For instance, in vision-based problems, the transformation between two images belongs to the LG $SE(3)$. In navigation-based problems, the unknown attitude of a mobile lies on $SO(3)$. On the other hand, in several target tracking applications, the parameters of interest belong to the manifold of Hermitian positive definite matrices [10], which can be equipped with a structure of LG. The associated lower bound can be obtained by minimizing an intrinsic MSE (IMSE) preserving the geometrical structure of the LG. In particular, in [9] [11], it is proposed an inequality on the intrinsic MSE for LGs, and provided Cramér-Rao bound on LGs (ICRB) for Euclidean observations with closed-form for the LG $SO(3)$. In [12], this bound has been formulated according to the Barankin and McAulay-Seidman formalism [13] [14] [15] and extended for the case where the observations can also lie on LGs. Especially, it is deduced an intrinsic Slepian-Bangs (ISB) formula for Gaussian model on LGs. It is relevant in some applications, for instance, when sensors provides orientation measurements such as odometer [16] or LIDAR system [17]. Most of problem of estimation on LGs assume that the uncertainty on observations is known. This hypothesis is not true in several real-world applications. For instance, in GNSS navigation, the Real-Time Kinematic (RTK) positioning uses observations of phase and distance in order to estimate the attitude in the LG $SO(3)$ of mobile equipped with several antennas. In this context, the noise measurements covariance matrix can be difficult to compute due to the correlations between observations [18]. Also, in computer vision, a camera provides noisy pixel detections depending on the unknown camera pose in $SE(3)$ with noise potentially unknown because linked to the signal to noise ratio of the observed image [19]. For all these problems, it is fundamental to obtain a lower bound that takes into account both unknown LG elements and unknown covariance noise. In this communication, we propose a generalization of the ISB formula introduced in [12], in the case where the variance for LG observation models is incorporated into the set of the unknown parameters. To achieve that, we consider that the covariance matrix is assumed equal to an identity matrix and multiply by an unknown scalar variance. In this case, we can obtain a unified formalism on LGs by using the intrinsic properties of the variance. Indeed, it is intrinsically constrained to lie on the LG of positive real values $\mathbb{R}^+$. Consequently, the unknown augmented parameter belongs to the augmented LG $G \times \mathbb{R}^+$. The ISB formula obtained is then computed for Gaussian models with observations on LG and for any augmented LG $G \times \mathbb{R}^+$. A closed-form is established for $SE(3) \times \mathbb{R}^+$ and for observations lying on $\mathbb{R}^3$: the resulting ICRB is numerically tested by comparison with the IMSE.

The communication is organized as follows: in the section II, we focus on the necessary background on LGs. In the section III, we remind the state-of-the-art ISB formula and we derive the proposed ISB for unknown variance parameter. Then, in the section IV, we obtain closed-form expression validated by numerical simulations.

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II. Background on LGs

In this section, we revisit the properties of LG and introduce the Lie group structure associated with $\mathbb{R}^+$.

A. Definition

A matrix space, denoted as LG, is a subset of $\mathbb{R}^{n\times n}$ endowed with the dual structure of a smooth manifold and a group. The group structure imparts an internal operation (matrix multiplication) while the smooth manifold structure defines a tangent space at each point of G. The identity tangent space, denoted as $\mathfrak{g}$, serves as the Lie algebra, where each element is locally connected to elements of G through the logarithmic and exponential mappings. These mappings, denoted as $\mathrm{Exp}_G : \mathfrak{g} \to G$ and $\mathrm{Log}_G : G \mapsto \mathfrak{g}$, are illustrated in Figure 1.

Since $\mathfrak{g}$ is isomorphic to $\mathbb{R}^m$, two bijections are defined: $[.]^\wedge : \mathbb{R}^m \mapsto \mathfrak{g}$ and $[.]^\vee : \mathfrak{g} \mapsto \mathbb{R}^m$, where m is the dimension of the Lie algebra.

Expressing the exponential and logarithmic mappings, for any $\mathbf{a} \in \mathbb{R}^m$, we have $\mathrm{Exp}_G^\wedge(\mathbf{a}) = \mathrm{Exp}([\mathbf{a}]_G^\wedge)$. Similarly, for any $\mathbf{M} \in G$, $[\mathrm{Log}_G(\mathbf{M})]_G^\vee = \mathrm{Log}_G^\vee(\mathbf{M})$.

For a deeper understanding of LG theory, interested readers can explore [20], [21].

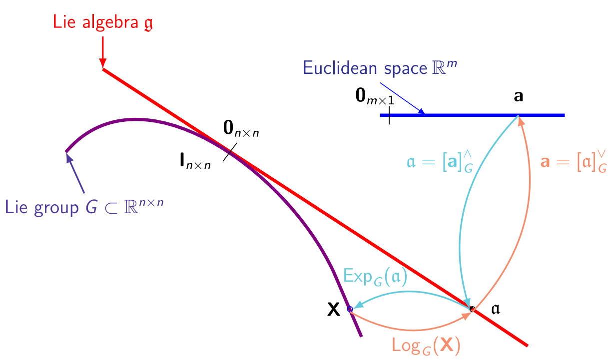

Fig. 1. Relation between $\mathbb{R}^m$, G and $\mathfrak{g}$.

B. Lie group $G \times \mathbb{R}^+$

The space $\mathbb{R}^+$ is a commutative LG equipped with the classical multiplication law and neutral element 1. Its logarithm and exponential application are given by the classical logarithm $\log(.)$ and exponential function $\exp(.)$ respectively defined on $\mathbb{R}^+$ and $\mathbb{R}$.
Let us consider an arbitrary matrix LG $G \subset \mathbb{R}^{n\times n}$ of dimension p, equipped with the classic matrix multiplication, and with the exponential and logarithm operator $\mathrm{Exp}_G^\wedge(.)$ and $\mathrm{Log}_G^\vee(.)$. Then, $G \times \mathbb{R}^+$ is also a LG with matrix multiplication where every element $\mathbf{X}^{(a)} \in G \times \mathbb{R}^+$ is written as:

$$\mathbf{X}^{(a)} = \begin{bmatrix} \mathbf{X} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & s \end{bmatrix} \ \forall \mathbf{X} \in G, \forall s \in \mathbb{R}^+, \tag{1}$$

and associated to the Euclidean element $\mathbf{a}^{(a)} = [\mathbf{a}, a_s] \in \mathbb{R}^p \times \mathbb{R}$. Thus, the logarithm and exponential operators on $G \times \mathbb{R}^+$ are defined by

$$\mathrm{Log}_{G\times\mathbb{R}^+}^\vee\left(\mathbf{X}^{(a)}\right) = \left[\mathrm{Log}_G^\vee(\mathbf{X})^\top, \log(s)\right]^\top, \tag{2}$$

$$\mathrm{Exp}_{G\times\mathbb{R}^+}^\wedge\left(\mathbf{a}^{(a)}\right) = \begin{bmatrix} \mathrm{Exp}_G^\wedge(\mathbf{a}) & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \exp(a_s) \end{bmatrix}. \tag{3}$$

III. Proposed Intrinsic Slepian-Bangs formula

A. Remind of the intrinsic Slepian-Bangs formula

Let us assume a set of independent observations $\mathbf{Z} = \{\mathbf{Z}_1, \ldots, \mathbf{Z}_N\}$ belonging to some LG G' (with dimension p') and following a concentrated Gaussian distribution [22]:

$$\mathbf{Z}_i = \mathbf{F}_i(\mathbf{X})\,\mathrm{Exp}_{G'}^\wedge(\mathbf{n}_i) \quad \mathbf{n}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}) \ \forall i \in \{1, \ldots, N\}, \tag{4}$$

where $\mathbf{X}$ is the unknown parameter belonging to G (with dimension p) and $\mathbf{F} : G \to G'$ is a smooth function. The IMSE, between an unbiased estimator $\widehat{\mathbf{X}}$ and $\mathbf{X}$, in the intrinsic sense [12], is given by

$$\mathbf{E} = \mathbb{E}\left(\mathrm{Log}_G^\vee\left(\mathbf{X}^{-1}\widehat{\mathbf{X}}\right)\mathrm{Log}_G^\vee\left(\mathbf{X}^{-1}\widehat{\mathbf{X}}\right)^\top\right) \tag{5}$$

is bounded by

$$\mathbf{P}_{ICRB} = \mathcal{J}^{-1}, \tag{6}$$

with $\mathcal{J}$ defined by the ISB formula [12]

$$\begin{aligned} \mathcal{J} &= -\mathbb{E}\left(\frac{\partial^2 \log p\left(\mathbf{Z}|\mathbf{X}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\right)}{\partial\boldsymbol{\delta}_1\partial\boldsymbol{\delta}_2}\right)\Bigg|_{\boldsymbol{\delta}_1,\boldsymbol{\delta}_2=\mathbf{0}} \\ &= \sum_{i=1}^N \mathcal{L}_{\mathbf{F}_i(\mathbf{X})}^{R}{}^\top\ \mathbb{E}\left(\tilde{\boldsymbol{\psi}}_i^\top \mathbf{\Sigma}^{-1}\,\tilde{\boldsymbol{\psi}}_i\right)\mathcal{L}_{\mathbf{F}_i(\mathbf{X})}^R, \end{aligned} \tag{7}$$

where $\tilde{\boldsymbol{\psi}}_i = \boldsymbol{\psi}_{G'}(\mathrm{Log}_{G'}^\vee\left(\mathbf{F}_i(\mathbf{X})^{-1}\mathbf{Z}_i\right))$ is the inverse of the left Jacobian of G' and $\mathcal{L}_{\mathbf{F}_i(\mathbf{X})}^R$ is the right Lie derivative of $\mathbf{F}_i$. For more informations on the expression of these quantities, the lector can refer to [23].

B. Intrinsic Slepian-Bangs formula on $G \times \mathbb{R}^+$

Now, consider the same model when $\mathbf{\Sigma} = \sigma^2\mathbf{I}_{p'\times p'}$ and σ^2 is unknown. As $\sigma^2 > 0$, it belongs to the LG $\mathbb{R}^+$, and the unknown parameter $\mathbf{X}^{(a)} \in G^{(a)} = G \times \mathbb{R}^+$

$$\mathbf{X}^{(a)} = \begin{bmatrix} \mathbf{X} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \sigma^2 \end{bmatrix}. \tag{8}$$

Theorem 1. *The IMSE $\mathbf{E}^{(a)}$ on $G^{(a)}$ between an unbiased estimator $\widehat{\mathbf{X}}^{(a)}$ and $\mathbf{X}^{(a)}$ verifies:*

$$\mathbf{E}^{(a)} \succeq \mathbf{P}_{ICRB}^{(a)} = \left(\mathcal{J}^{(a)}\right)^{-1}$$

$$\mathcal{J}^{(a)} = \begin{bmatrix} \mathcal{J}_X & \mathcal{J}_{X,s} \\ \mathcal{J}_{X,s}^\top & \mathcal{J}_s \end{bmatrix}$$

with $\forall \boldsymbol{\epsilon}_1^{\mathbf{X}} \in \mathbb{R}^p, \boldsymbol{\epsilon}_2^{\mathbf{X}} \in \mathbb{R}^p, \epsilon_1^{\sigma} \in \mathbb{R}, \epsilon_2^{\sigma} \in \mathbb{R}$

$$\mathcal{J}_X = -\mathbb{E}\left(\frac{\partial^2 \log p \left(\mathbf{Z} | \mathbf{X}^{(a)} \operatorname{Exp}_{G^{(a)}}^{\wedge}\left(\begin{bmatrix} \boldsymbol{\epsilon}_1^{\mathbf{X}} \\ 0 \end{bmatrix} \right) \ \operatorname{Exp}_{G^{(a)}}^{\wedge}\left(\begin{bmatrix} \boldsymbol{\epsilon}_2^{\mathbf{X}} \\ 0 \end{bmatrix} \right) \right)}{\partial \boldsymbol{\epsilon}_1^{\mathbf{X}} \partial \boldsymbol{\epsilon}_2^{\mathbf{X}}} \right) \Bigg|_{\boldsymbol{\epsilon}_1^{\mathbf{X}}, \boldsymbol{\epsilon}_2^{\mathbf{X}} = \mathbf{0}} = \mathcal{J} \tag{9}$$

$$\mathcal{J}_s = -\mathbb{E}\left(\frac{\partial^2 \log p \left(\mathbf{Z} | \mathbf{X}^{(a)} \operatorname{Exp}_{G^{(a)}}^{\wedge}\left(\begin{bmatrix} \mathbf{0} \\ \epsilon_1^{\sigma} \end{bmatrix} \right) \ \operatorname{Exp}_{G^{(a)}}^{\wedge}\left(\begin{bmatrix} \mathbf{0} \\ \epsilon_2^{\sigma} \end{bmatrix} \right) \right)}{\partial \epsilon_1^{\sigma} \partial \epsilon_2^{\sigma}} \right) \Bigg|_{\epsilon_1^{\sigma}, \epsilon_2^{\sigma} = 0} = \frac{p' N}{2} \tag{10}$$

$$\mathcal{J}_{X,s} = \frac{1}{\sigma^2} \mathbb{E}\left(\left(\mathcal{L}_{\mathbf{f}_i(\mathbf{X})}^{R} \right)^{\top} \boldsymbol{\psi}_{G'}(\operatorname{Log}_{G'}^{\vee} \left(\mathbf{F}_i(\mathbf{X})^{-1} \mathbf{Z}_i \right))^{\top} \operatorname{Log}_{G'}^{\vee} \left(\mathbf{F}_i(\mathbf{X})^{-1} \mathbf{Z}_i \right) \right) \tag{11}$$

Proof:

To begin, let us define two notations:

$$lp(\mathbf{M}, \boldsymbol{\epsilon}) = \log p(\mathbf{Z} | \mathbf{X} \operatorname{Exp}_G^{\wedge}(\boldsymbol{\epsilon})) \tag{12}$$

$$lp(\mathbf{X}, \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) = \log p(\mathbf{Z} | \mathbf{X} \operatorname{Exp}_G^{\wedge}(\boldsymbol{\epsilon}_1) \operatorname{Exp}_G^{\wedge}(\boldsymbol{\epsilon}_2)) \tag{13}$$

As the unknown parameters is divided into two parts, we can decompose $\mathcal{J}^{(a)}$ as follows:

$$\mathcal{J}^{(a)} = \begin{bmatrix} \mathcal{J}_X & \mathcal{J}_{X,s} \\ \mathcal{J}_{X,s}^{\top} & \mathcal{J}_s \end{bmatrix} \tag{14}$$

with:

$$\mathcal{J}_X = \mathbb{E}\left(\frac{\partial lp(\mathbf{X}, \left[\boldsymbol{\epsilon}_1^{\mathbf{X}}; 0\right])}{\partial \boldsymbol{\epsilon}_1^{\mathbf{X}}} \frac{\partial lp(\mathbf{X}, \left[\boldsymbol{\epsilon}_2^{\mathbf{X}}; 0\right])}{\partial \boldsymbol{\epsilon}_2^{\mathbf{X}}}^{\top} \Bigg|_{\boldsymbol{\epsilon}_1^{\mathbf{X}}, \boldsymbol{\epsilon}_2^{\mathbf{X}} = \mathbf{0}} \right) \tag{15}$$

$$\mathcal{J}_s = \mathbb{E}\left(\frac{\partial lp(\mathbf{X}, [\mathbf{0}; \epsilon_1^{\sigma}])}{\partial \epsilon_1^{\sigma}} \frac{\partial lp(\mathbf{X}, [\mathbf{0}; \epsilon_2^{\sigma}])}{\partial \epsilon_2^{\sigma}}^{\top} \Bigg|_{\epsilon_1^{\sigma}, \epsilon_2^{\sigma} = 0} \right) \tag{16}$$

$$\mathcal{J}_{X,s} = \mathbb{E}\left(\frac{\partial lp(\mathbf{X}, \left[\boldsymbol{\epsilon}_1^{\mathbf{X}}; 0\right])}{\partial \boldsymbol{\epsilon}_1^{\mathbf{M}}} \frac{\partial lp(\mathbf{X}, [\mathbf{0}; \epsilon_2^{\sigma}])}{\partial \epsilon_2^{\sigma}}^{\top} \Bigg|_{\boldsymbol{\epsilon}_1^{\mathbf{X}} = \mathbf{0}, \epsilon_2^{\sigma} = 0} \right) \tag{17}$$

• Computation of $\mathcal{J}_X$:

By applying the following Slepian-Bangs formula on $\mathbf{X}$, we obtain directly:

$$\mathcal{J}_X = \tag{18}$$

$$\frac{1}{\sigma^2} \sum_{i=1}^{N} \left(\mathcal{L}_{\mathbf{f}_i(\mathbf{X})}^{R} \right)^{\top} \mathbb{E}\left(\boldsymbol{\psi}_{G'}(\operatorname{Log}_{G'}^{\vee} \left(\mathbf{F}_i(\mathbf{X})^{-1} \mathbf{Z}_i \right)) \times \boldsymbol{\psi}_{G'}(\operatorname{Log}_{G'}^{\vee} \left(\mathbf{F}_i(\mathbf{X})^{-1} \mathbf{Z}_i \right))^{\top} \right) \mathcal{L}_{\mathbf{f}_i(\mathbf{X})}^{R} \tag{19}$$

• Computation of $\mathcal{J}_s$:

As $\boldsymbol{\epsilon} \to \log p \left(\mathbf{Z} | \mathbf{X} \operatorname{Exp}_G^{\wedge}(\boldsymbol{\epsilon}) \right)$ is a quadratic function, we can consider it sufficiently regular so that:

$$\mathbb{E}\left(\frac{\partial lp(\mathbf{X}, [\mathbf{0}; \epsilon_1^{\sigma}])}{\partial \epsilon_1^{\sigma}} \frac{\partial lp(\mathbf{X}, [\mathbf{0}; \epsilon_2^{\sigma}])}{\partial \epsilon_2^{\sigma}}^{\top} \Bigg|_{\epsilon_1^{\sigma}, \epsilon_2^{\sigma} = 0} \right) = -\mathbb{E}\left(\frac{\partial lp^2(\mathbf{X}, [\mathbf{0}; \epsilon_1^{\sigma}], [\mathbf{0}; \epsilon_2^{\sigma}])}{\partial \epsilon_1^{\sigma} \partial \epsilon_2^{\sigma}} \Bigg|_{\epsilon_1^{\sigma}, \epsilon_2^{\sigma} = 0} \right) \tag{20}$$

On the other hand, we know that:

$$\frac{\partial^2 lp(\mathbf{X}, [\mathbf{0}; \epsilon_1^{\sigma}], [\mathbf{0}; \boldsymbol{\epsilon}_2^{\sigma}])}{\partial \epsilon_1^{\sigma} \, \partial \epsilon_2^{\sigma}} \Bigg|_{\epsilon_1^{\sigma} = \epsilon_2^{\sigma} = 0} = \frac{\partial^2}{\partial \epsilon_1^{\sigma} \, \partial \epsilon_2^{\sigma}} \left(-\frac{N}{2} \log \left(\sigma^2 \exp(\epsilon_1^{\sigma}) \exp(\epsilon_2^{\sigma}) \right) - \frac{1}{2} \sum_{i=1}^{N} \operatorname{Log}_{G'}^{\vee} \left(\mathbf{F}_i(\mathbf{X})^{-1} \mathbf{Z}_i \right)^{\top} (\sigma^2 \exp(\epsilon_1^{\sigma}) \exp(\epsilon_2^{\sigma}))^{-1} \operatorname{Log}_{G'}^{\vee} \left(\mathbf{F}_i(\mathbf{X})^{-1} \mathbf{Z}_i \right) \tag{21}$$

and can be simplified in the following way:

- first, it is straightforward that:

$$\frac{\partial^2}{\partial \epsilon_1^{\sigma} \, \partial \epsilon_2^{\sigma}} \log \left(\sigma^2 \exp(\epsilon_1^{\sigma}) \exp(\epsilon_2^{\sigma}) \right) \Bigg|_{\epsilon_1^{\sigma}, \epsilon_2^{\sigma} = 0} = 0 \tag{22}$$

- second, we show that:

$$\frac{\partial^2}{\partial \epsilon_1^{\sigma} \, \partial \epsilon_2^{\sigma}} (\sigma^2 \exp(\epsilon_1^{\sigma}) \exp(\epsilon_2^{\sigma}))^{-1} \Bigg|_{\epsilon_1^{\sigma}, \epsilon_2^{\sigma} = 0} = \frac{1}{\sigma^2} \tag{23}$$

Consequently, we obtain:

$$\frac{\partial^2 lp(\mathbf{X}, [\mathbf{0}; \epsilon_1^{\sigma}], [\mathbf{0}; \epsilon_2^{\sigma}])}{\partial \epsilon_1^{\sigma} \, \partial \epsilon_2^{\sigma}} \Bigg|_{\epsilon_1^{\sigma} = 0, \epsilon_2^{\sigma} = 0} = -\frac{1}{2\,\sigma^2} \sum_{i=1}^{N} \operatorname{tr}\left(\operatorname{Log}_{G'}^{\vee} \left(\mathbf{F}_i(\mathbf{X})^{-1} \mathbf{Z}_i \right)^{\top} \operatorname{Log}_{G'}^{\vee} \left(\mathbf{F}_i(\mathbf{X})^{-1} \mathbf{Z}_i \right) \right) \tag{24}$$

and:

$$\mathbb{E}\left(\frac{\partial^2 lp(\mathbf{X}, [\mathbf{0}; \epsilon_1^{\sigma}], [\mathbf{0}; \epsilon_2^{\sigma}])}{\partial \epsilon_1^{\sigma} \, \partial \epsilon_2^{\sigma}} \Bigg|_{\epsilon_1^{\sigma} = \epsilon_2^{\sigma} = 0} \right) = -\frac{1}{2\,\sigma^2} \sum_{i=1}^{N} \operatorname{tr}(\mathbb{E}(\operatorname{Log}_{G'}^{\vee} \left(\mathbf{F}_i(\mathbf{X})^{-1} \mathbf{Z}_i \right) \operatorname{Log}_{G'}^{\vee} \left(\mathbf{F}_i(\mathbf{X})^{-1} \mathbf{Z}_i \right)^{\top})$$

As:

$$\mathbb{E}\left(\mathrm{Log}^{\vee}_{G'}\left(\mathbf{F}_i(\mathbf{X})^{-1}\mathbf{Z}_i\right)\ \mathrm{Log}^{\vee}_{G'}\left(\mathbf{F}_i(\mathbf{X})^{-1}\mathbf{Z}_i\right)^{\top}\right)=\sigma^2\mathbf{I}; \tag{25}$$

we finally obtain that $\mathcal{J}_s=\dfrac{p'\,N}{2}$.

• <u>Computation of $\mathcal{J}_{X,s}$:</u>

In the same way as previously, if $\boldsymbol{\epsilon} \to \log p\left(\mathbf{Z}|\mathbf{X}\,\mathrm{Exp}^{\wedge}_{G}(\boldsymbol{\epsilon})\right)$ is sufficiently regular, then:

$$\mathbb{E}\left(\frac{\partial lp(\mathbf{X},[\boldsymbol{\epsilon}^{\mathbf{X}}_1;0])}{\partial \boldsymbol{\epsilon}^{\mathbf{X}}_1}\,\frac{\partial lp(\mathbf{X},[\mathbf{0};\epsilon^{\sigma}_2])}{\partial \epsilon^{\sigma}_2}^{\top}\Bigg|_{\boldsymbol{\epsilon}^{\mathbf{X}}_1,\epsilon^{\sigma}_2=0}\right)= -\,\mathbb{E}\left(\frac{\partial^2 lp\left(\mathbf{X},[\boldsymbol{\epsilon}^{\mathbf{X}}_1;0],[\mathbf{0};\epsilon^{\sigma}_2]\right)}{\partial \boldsymbol{\epsilon}^{\mathbf{X}}_1\,\partial \epsilon^{\sigma}_2}\Bigg|_{\boldsymbol{\epsilon}^{\mathbf{X}}_1,\epsilon^{\sigma}_2=0}\right) \tag{26}$$

Furthermore, as:

$$\frac{\partial \mathrm{Log}^{\vee}_{G'}\left(\mathbf{F}_i(\mathbf{X}\,\mathrm{Exp}^{\wedge}_{G}(\boldsymbol{\epsilon}))^{-1}\mathbf{Z}_i\right)}{\partial \boldsymbol{\epsilon}}\Bigg|_{\boldsymbol{\epsilon}=\mathbf{0}}= \psi'_G(\mathrm{Log}^{\vee}_{G'}\left(\mathbf{F}_i(\mathbf{X})^{-1}\mathbf{Z}_i\right)\ \mathcal{L}^{R}_{\mathbf{F}_i(\mathbf{X})} \tag{27}$$

we can demonstrate that:

$$\frac{\partial lp(\mathbf{X},[\boldsymbol{\epsilon}^{\mathbf{X}}_1;0])}{\partial \boldsymbol{\epsilon}^{\mathbf{X}}_1}=\frac{1}{\sigma^2}\left(\mathcal{L}^{R}_{\mathbf{F}_i(\mathbf{X})}\right)^{\top}\times \psi_{G'}(\mathrm{Log}^{\vee}_{G'}\left(\mathbf{F}_i(\mathbf{X})^{-1}\mathbf{Z}_i\right))^{\top}\,\mathrm{Log}^{\vee}_{G'}\left(\mathbf{F}_i(\mathbf{X}\,\mathrm{Exp}^{\wedge}_{G}\left(\boldsymbol{\epsilon}^{\mathbf{X}}_1\right))^{-1}\mathbf{Z}_i\right) \tag{28}$$

Therefore, by differentiating the previous expression according to ϵ^{σ}_2, we gather:

$$\frac{\partial^2 lp(\mathbf{X},[\boldsymbol{\epsilon}^{\mathbf{X}}_1;0],[\mathbf{0};\epsilon^{\sigma}_2])}{\partial \boldsymbol{\epsilon}^{\mathbf{X}}_1\,\partial \epsilon^{\sigma}_2}=\left(\mathcal{L}^{R}_{\mathbf{F}_i(\mathbf{X})}\right)^{\top}\frac{\partial\left(\sigma^2\,\exp\left(\epsilon^{\sigma}_2\right)\right)^{-1}}{\partial \epsilon^{\sigma}_2}\ \mathrm{Log}^{\vee}_{G'}\left(\mathbf{F}_i(\mathbf{X}\,\mathrm{Exp}^{\wedge}_{G}\left(\boldsymbol{\epsilon}^{\mathbf{X}}_1\right))^{-1}\mathbf{Z}_i\right) \tag{29}$$

By using the fact that:

$$\frac{\partial\left(\sigma^2\,\exp\left(\epsilon^{\sigma}_2\right)\right)^{-1}}{\partial \epsilon^{\sigma}_2}\Bigg|_{\epsilon^{\sigma}_2=0}=\frac{1}{\sigma^2}, \tag{30}$$

we deduce that:

$$\frac{\partial^2 lp\left(\mathbf{X},[\boldsymbol{\epsilon}^{\mathbf{X}}_1;0],[\mathbf{0};\epsilon^{\sigma}_2]\right)}{\partial \boldsymbol{\epsilon}^{\mathbf{X}}_1\,\partial \epsilon^{\sigma}_2}\Bigg|_{\boldsymbol{\epsilon}^{\mathbf{X}}_1=\mathbf{0},\epsilon^{\sigma}_2=0}= -\frac{1}{\sigma^2}\left(\mathcal{L}^{R}_{\mathbf{F}_i(\mathbf{X})}\right)^{\top}\psi_{G'}(\mathrm{Log}^{\vee}_{G'}\left(\mathbf{F}_i(\mathbf{X})^{-1}\mathbf{Z}_i\right))^{\top}\,\mathrm{Log}^{\vee}_{G'}\left(\mathbf{F}_i(\mathbf{X})^{-1}\mathbf{Z}_i\right) \tag{31}$$

Then, by taking the expected value, we yield:

$$\mathbb{E}\left(\frac{\partial^2 lp\left(\mathbf{X},\boldsymbol{\epsilon}^{\mathbf{X}}_1,\epsilon^{\sigma}_2\right)}{\partial \boldsymbol{\epsilon}^{\mathbf{X}}_1\,\partial \epsilon^{\sigma}_2}\right)\Bigg|_{\boldsymbol{\epsilon}^{\mathbf{X}}_1=\mathbf{0},\epsilon^{\sigma}_2=0}=-\frac{1}{\sigma^2}\times \mathbb{E}\left(\left(\mathcal{L}^{R}_{\mathbf{F}_i(\mathbf{X})}\right)^{\top}\psi_{G'}(\mathrm{Log}^{\vee}_{G'}\left(\mathbf{F}_i(\mathbf{X})^{-1}\mathbf{Z}_i\right))^{\top}\,\mathrm{Log}^{\vee}_{G'}\left(\mathbf{F}_i(\mathbf{X})^{-1}\mathbf{Z}_i\right)\right) \tag{32}$$

Note that $\mathcal{J}_s$ is different the information matrix of σ^2 in the case where it is treated as an Euclidean parameter in $\mathbb{R}$. It is noteworthy that the well known result for the standard conditional signal model [1-5], that is, an asymptotic uncoupling between the estimation of the noise power σ^2 and the remaining parameters, is still valid for LGs.

IV. Simulation results

To illustrate the proposed bound, we consider a set of Euclidean observations $\{\mathbf{z}_i\}_{i=1}^{N}$ described by the following model:

$$\mathbf{z}_i=\mathbf{R}\mathbf{p}_i+\mathbf{p}+\mathbf{n}_i\quad \mathbf{n}_i\sim\mathcal{N}(\mathbf{0},\sigma^2\mathbf{I}_{3\times3}), \tag{33}$$

where $\mathbf{p}_i\in\mathbb{R}^3$. $\mathbf{R}\in SO(3)$ and $\mathbf{p}\in\mathbb{R}^3$ are respectively a rotation and a translation parameter. It can be written under the compact form:

$$\mathbf{z}_i=\mathbf{\Pi}\,\mathbf{X}\left[\mathbf{p}_i^{\top},1\right]^{\top}+\mathbf{n}_i. \tag{34}$$

with $\mathbf{X}=\begin{bmatrix}\mathbf{R} & \mathbf{p}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\in G=SE(3)$ and can be seen as an Euclidean simplification of the model (4) where $G'=\mathbb{R}^3$, $\mathbf{Z}_i=\mathbf{z}_i$, $\mathbf{F}_i(\mathbf{X})=\mathbf{\Pi}\,\mathbf{X}\left[\mathbf{p}_i^{\top},1\right]^{\top}$, $\psi_{G'}(.)=\mathbf{I}_{3\times3}$. In this case, the ISB formula on $\mathbf{X}^{(a)}$ is given by:

$$\mathcal{J}^{(a)}=\begin{bmatrix}\dfrac{1}{\sigma^2}\sum\limits_{i=1}^{N}\mathbf{J}^{(i)}_X\,\mathbf{J}^{(i)\,\top}_X & \mathbf{0}_{p\times1}\\ \mathbf{0}_{1\times p} & \dfrac{3\,N}{2}\end{bmatrix}, \tag{35}$$

with

$$\mathbf{J}^{(i)}_X=\left[\mathbf{\Pi}\,\mathbf{X}\,\mathbf{G}_1\,\mathbf{p}_i,\ldots,\mathbf{\Pi}\,\mathbf{X}\,\mathbf{G}_6\,\mathbf{p}_i\right], \tag{36}$$

and $\{\mathbf{G}_l\}_{l=1}^{6}$ a basis of the Lie algebra of $SE(3)$.

In order to illustrate numerically the proposed ICRB, the bound associated to (13) is simulated with two values of $\sigma^2=0.1$ and $\sigma^2=0.6$. The points $\{\mathbf{p}_i\}_{i=1}^{N}$ are generated with the following formula:

$$\mathbf{p}_i\sim\mathcal{N}_{\mathbb{R}^3}(\mathbf{p}_m,\sigma_m^2\,\mathbf{I}_{3\times3})\ \forall i\in\{1,\ldots N\}, \tag{37}$$

where $\mathbf{p}_m=[1,1,1]^{\top}$ and $\sigma_m=0.1$. In order to evaluate the trace of the IMSE, we approximate it by Monte Carlo,

$$\frac{1}{N_{mc}}\sum_{i=1}^{N_{mc}}\left\|\mathrm{Log}^{\vee}_{G'}\left(\mathbf{X}^{(a)\,-1}\left(\widehat{\mathbf{X}}^{(a)}\right)_i\right)\right\|^2,$$

where N_{mc} is the number of realizations of the algorithm and $\left(\widehat{\mathbf{X}}^{(a)}\right)_i$ the estimator for the i-th realization in the ML sense. Such estimator is built with a Gauss-Newton algorithm dedicated to LGs [24] and σ^2 is estimated at each iteration l by its empirical estimator:

$$\sigma^{2\,(l)}=\frac{1}{N-1}\sum_{i=1}^{N}\|\mathbf{z}_i-\mathbf{\Pi}\,\mathbf{X}^{(l)}\left[\mathbf{p}_i^{\top},1\right]^{\top}\|^2, \tag{38}$$

where $\mathbf{X}^{(l)}$ is the estimation of $\mathbf{X}$ at iteration l.

In Figures 1 and 2, we draw the evolution of the trace of the ICRB superimposed to the trace of the IMSE as function of the number of observations. We remark that the latter converges to the proposed ICRB which shows the consistency of the latter. Furthermore, we observe that the convergence is faster when $\sigma^2 = 0.1^2$ than $\sigma^2 = 0.6^2$, which is a relevant behaviour in line with the well-known behaviour of the MLE in the the Gaussian Euclidean case.

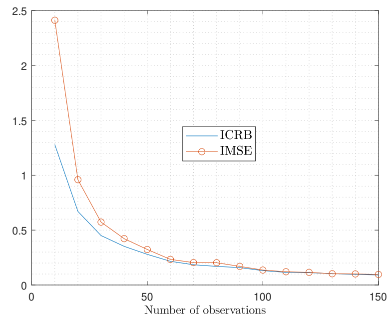

Fig. 2. Evolution of the proposed IMSB and IMSE for $\sigma^2 = 0.1^2$ ($N_{mc} = 1000$)

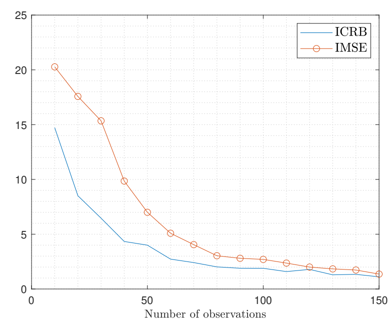

Fig. 3. Evolution of the proposed IMSB and IMSE for $\sigma^2 = 0.6^2$.($N_{mc} = 1000$))

V. Conclusions

In this article, we have derived an intrinsic Slepian-Bangs formula applicable to Lie groups, particularly when dealing with estimation problems involving unknown diagonal covariance matrices. The establishment of this bound involved harnessing the inherent Lie group structure of the covariance matrix space, leading to the identification of a parameter within the product of two Lie groups. Through this approach, we formulated a closed-form expression, for the Wahba's problem, commonly used in signal processing problems. The implications of this work are extensive. Firstly, a relevant perspective for further exploration involves extending to the case where the covariance matrix is full. Secondly, a challenging yet pivotal endeavor is the adaptation of this framework for dynamic parameters. This is particularly crucial in the context of tracking problems associated with Lie groups, where the covariance matrix of the process model remains unknown.

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